Journal of Statistical Physics, Vol. 19, No. 4, 1978

Susceptibility of the Rectangular Ising Ferromagnet

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Received January 16, 1978

The critical index values $\gamma = 7/4$ for the susceptibility and $\delta = 15$ for the critical isotherm are derived rigorously for the rectangular Ising ferromagnet with nearest neighbor interactions. The critical indices associated with the Fisher moment definition of the correlation length are obtained as $T \to T_c+$. The index of the fluctuation sum definition of critical correlations is obtained.

KEY WORDS: Ising model; lattice statistic; two-dimensional systems; critical indices; transfer matrix; correlation length.

1. INTRODUCTION

Some recent progress in the theory of the rectangular Ising ferromagnet and the associated transfer matrix[(1)] has led to the analysis of the interface profile in the two-phase region.[(2)] As was reported briefly,[(3)] the same method gave the first *rigorous* derivation of the critical exponents γ and γ' for the divergence of the magnetic susceptibility; this paper enlarges upon and extensively simplifies the derivation of the result $\gamma = 7/4$. Since then Barouch, McCoy, and co-workers[(4)] have evaluated the amplitudes associated with the divergences by passing in a heuristic way to what is known as the scaling limit. This process has now been accomplished with full rigor for $T > T_c$: it is not needed, however, in the approach reported here to determine γ.

The Ising model on a toroidal lattice is defined by specifying the energy of a configuration $\{\sigma\}$ of spins $\sigma_{\mathbf{i}} = \pm 1$ located at each vertex $\mathbf{i}$ of Λ, which is a subset of $\mathbb{Z}^2$ of the form

$$\Lambda = \{\mathbf{i}\colon\ 1 \leqslant i_1 \leqslant N,\quad 1 \leqslant i_2 \leqslant M\} \tag{1}$$

The energy $E_\Lambda(\{\sigma\})$ is given by

$$E_\Lambda(\{\sigma\}) = -J_1 \sum_{\mathbf{i}\in\Lambda} \sigma_{i_1 i_2}\sigma_{i_1+1,i_2} - J_2 \sum_{\mathbf{i}\in\Lambda} \sigma_{i_1 i_2}\sigma_{i_1,i_2+1} - H \sum_{\mathbf{i}\in\Lambda} \sigma_{\mathbf{i}} \tag{2}$$

Partially supported by grant PHY 76 17191.

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In the above H is a magnetic field and $J_1, J_2 > 0$ are nearest neighbor ferromagnetic couplings. The canonical probability measure on the phase space $\{-1, 1\}^{|\Lambda|}$ is

$$p_\Lambda(\{\sigma\}) = Z_\Lambda^{-1} \exp[-\beta E_\Lambda(\{\sigma\})] \tag{3}$$

with $\beta = 1/kT$. Henceforth the notations

$$h = \beta H, \qquad K_i = \beta J_i, \quad i = 1, 2 \tag{4}$$

will be used. The susceptibility is defined by

$$\chi(\beta, h) = \partial_h \lim_{\Lambda \to \infty} \langle \sigma_i \rangle_\Lambda \tag{5}$$

where $\langle \cdots \rangle_\Lambda$ denotes expectation with respect to (3). The limit $\Lambda \to \infty$ is to be taken in the sense of Van Hove. It is known that χ can be expressed as a sum of truncated correlation functions[5]

$$\chi(\beta, h) = 1 + \sum_{\mathbf{r} \neq 0} u_2(\mathbf{r}) \tag{6}$$

where

$$u_2(\mathbf{r}) = \lim_{\Lambda \to \infty} (\langle \sigma_0 \sigma_\mathbf{r} \rangle_\Lambda - \langle \sigma_0 \rangle_\Lambda{}^2) \tag{7}$$

This formula is true also for the two-phase region provided the state is extremal, i.e., one must have plus or minus boundary conditions.[5,6] Whenever $h \neq 0$, $\chi(\beta, h)$ is bounded, but on $h = 0$, there exists a critical temperature T_c defined by[7]

$$\sinh 2K_1(c) \sinh 2K_2(c) = 1 \tag{8}$$

at which χ diverges. Hereafter the notation $t = (T - T_c)/T_c$ will be used and all thermodynamic functions will have arguments (t, h). The critical exponents are defined by

$$\gamma \text{ (resp. } \gamma') = -\lim_{t \to 0+} \left(\text{resp.} \lim_{t \to 0-}\right) \log \chi(t, 0)/\log|t| \tag{9}$$

The procedure used here to investigate (9), besides being rigorous, relates to basic approximate theories. The Ornstein–Zernike theory of correlations, recently placed an exact basis,[8] states that the decay of $u_2(\mathbf{r})$ is given by two complex-conjugate simple poles on the imaginary axis in the transform $\hat{u}_2(|k|)$. In the transfer matrix approach, the spin operator scatters from the vacuum, or equilibrium state, to many-particle states. The one-particle states give the Ornstein–Zernike pole, but the residue has to be modified by a factor $|t|^{1/4}$. For a ferromagnet the transfer matrix has only nonnegative eigenvalues, so that, as will be seen later, the Ornstein–Zernike theory, with corrected residue, gives a lower bound to (7).

The intuitive idea behind the heuristic calculations of $\chi(0, t)$ by Fisher[9] and Kadanoff[10] is that only fluctuations on the length scale of the correlation length ξ make a significant contribution to χ. These fluctuations involve scattering to states with *any number* of particles and are thus not described fully by Ornstein–Zernike theory. Fisher's theory is correct as an upper bound to the fluctuations in (7) for large scaled lengths, whereas Kadanoff's result is an upper bound for small scaled lengths. The latter case is very hard to treat for $T < T_c$, but in this case can be handled quite simply by use of Griffiths' inequalities[15] and an ingenious application of duplication by Messager and Miracle-Sole[12] to exploit the exact result for $u_2((n, n)|0, 0)$ given by Onsager.[13] The upper and lower bound thus constructed both diverge as $|t|^{7/4}$, establishing that $\gamma = 7/4$.

The modification to the classical theory of Ornstein–Zernike is thus twofold; but it is a highly intriguing fact that correction of the residue of the pole, but neglect of the many-particle scattering, gives the amplitude of the divergence to better than 1%.

The correlation length is defined in terms of the Onsager function $\gamma(u)$ given by[11]

$$\cosh \gamma(u) = \cosh 2K_1{}^* \cosh 2K_2 - \sinh 2K_1{}^* \sinh 2K_2 \cos u \tag{10}$$

where $\gamma(u) \geqslant 0$ for real u defines the branch and $\exp(-2K_1{}^*) = \tanh K_1$. It can be shown[14] that $\xi(t, 0) = 1/\gamma(0)$. In terms of the scaled lengths $s = r\gamma(0)$ the sums in (6) might well be replaced by Riemann integrals; this is in fact the scaling limit. What one would like to be able to prove is that there exist suitable functions $F_\pm(s)$ such that

$$\lim_{t \to 0\pm} \chi(t, 0) t^{7/4} = \int_0^\infty sF_\pm(s)\, ds \tag{11}$$

where we have assumed angular independence of $u(\mathbf{r})$ in its scaled form.[3,8] Such results are quite crucial in the theory of critical phenomena. The methods used in this paper enable us to establish an analog of (11) for $s \geqslant s_0$, a constant. Considerable refinement is needed to go to (11) for $t \to 0-$.

Convexity arguments due to Griffiths[16] have established exponent inequalities for the critical isotherm: let

$$1/\delta = \lim_{h \to 0+} \log m(0, h)/\log h \tag{12}$$

where

$$m(t, h) = \lim_{\Lambda \to \infty} \langle \sigma_i \rangle_\Lambda \tag{13}$$

is the magnetization. Then, for the rectangular Ising ferromagnet which has $\alpha = \alpha' = 0$ (the specific heat exponents),[11] Griffiths has shown that (assuming δ exists)

$$(2 - \alpha)(\delta - 1) \leqslant \gamma(\delta + 1) \tag{14}$$

and that

$$\beta(1 + \delta) \geqslant 2 - \alpha' \tag{15}$$

In the above, β is the index for the spontaneous magnetization. In the present case, $\beta = 1/8$.[17] The result of this paper that $\gamma = 7/4$, taken with (14) and (15), then implies that $\delta = 15$. Considerably more recondite considerations enable one to prove that

$$15 \geqslant \delta^+ \geqslant \delta^- \geqslant 15 \tag{16}$$

where

$$\delta^+ \text{ (resp. } \delta^-) = \limsup_{h \to 0+} \text{(resp. inf)} \log m(0, h)/\log h \tag{17}$$

In this way the assumed existence of δ is established; this is somewhat arcane. The numerical results of Gaunt and Sykes[18] ($\delta = 15 \pm 0.008$) are in excellent agreement.

Finally, Fisher's moment definition of the correlation length and of the critical correlation exponent will be investigated.

2. TRANSFER MATRIX

The starting point of this calculation is the expression for $u(\mathbf{r})$ in terms of the transfer matrix spectrum and matrix elements in the basis generated by its eigenvectors. In the limit $N, M \to \infty$, it has been shown that[1] for $t > 0$, $h = 0$,

$$u(\mathbf{r}) = \sum_0^\infty \frac{1}{(2j + 1)!} \left(\frac{1}{2\pi}\right)^{2j+1} \int_{-\pi}^{\pi} \cdots \int d(\omega)_{2j+1} K((\omega)_{2j+1})$$
$$\times \exp - \sum_1^{2j+1} [|r_1|\gamma(\omega_k) + ir_2\omega_k] \tag{18}$$

where

$$K((\omega)_{2j+1}) = |F^x((e^{i\omega})_{2j+1})|^2 \tag{19}$$

with the standard notation $(\omega)_n = (\omega_1, \ldots, \omega_n)$. The following definition is useful:

$$\Delta_j(z)_n = (z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n), \qquad 1 \leqslant j \leqslant n \tag{20}$$

$$\Delta_I = \prod_{j \in I} \Delta_j \tag{21}$$

for any index set I.

Then

$$F^x((z)_{2n+1}) = \sum_{1}^{2n+1} (-1)^j g(z_j) F(\Delta_j(z)_{2n+1}) \tag{22}$$

with F defined recursively by

$$F((z)_{2n}) = \sum_{2}^{2n} (-1)^j f_-(z_1 z_j) F(\Delta_{1j}(z)_{2n}) \tag{23}$$

The boundary condition is

$$F(\phi) = [1 - (\sinh 2K_1 \sinh 2K_2)^2]^{1/8} \tag{24}$$

and

$$g(t) = (B/A)^{1/2} t / [(t - A^{-1})(t - B)]^{1/2} \tag{25}$$

$$f_-(zt) = \{f(z)[f(t^{-1})]^{-1} - f(t)[f(z^{-1})]^{-1}\} zt/(zt - 1)$$

with

$$f(z) = [(z - A)(z - B^{-1})]^{1/2} \tag{26}$$

and

$$A = \coth K_2 \coth K_1{}^*, \qquad B = \tanh K_2 \coth K_1{}^* \tag{27}$$

The above results are clearly of Pfaffian form.[19]

3. LOWER BOUNDS FOR χ

From (18) it follows that

$$u(\mathbf{r}) \leqslant \exp[-|r_1| \gamma(0)] \tag{28}$$

by use of the triangle inequality and completeness. Thus

$$\sum_{\mathbf{r} \neq 0} u_2(\mathbf{r}) \leqslant \sum_{0}^{\infty} n \exp[-n\gamma(0)] \tag{29}$$

which converges if $\gamma(0) > 0$. Thus the double sum implied by (6) and (18) converges absolutely and may be rearranged to give

$$\chi(t, 0) = 1 + \sum_{0}^{\infty} \chi_{2j+1}(t) \tag{30}$$

where

$$\chi_{2j+1}(t) = \frac{1}{(2j+1)!} \sum_{\mathbf{r}\neq 0} \left(\frac{1}{2\pi}\right)^{2j+1} \int_{-\pi}^{\pi} \cdots \int d(\omega)_{2j+1}$$
$$\times K(\omega)_{2j+1} \exp - \sum_{1}^{2j+1} [|r_1|\gamma(\omega_k) + ir_2\omega_k] \tag{31}$$

The sum is conveniently divided into the regions $R_1 = \{\mathbf{r}\colon 1 \leqslant r_1 < \infty, -r_1 + 1 \leqslant r_2 \leqslant r_1\}$ and its images R_j, $j = 2, 3, 4$, under rotation. Just the R_1 contribution will be given, since the others are trivially related to it.

By dominated convergence, the order of summation and integration in (31) can be reversed, giving

$$\chi_{2j+1}(t) = \frac{1}{(2j+1)!} \left(\frac{1}{2\pi}\right)^{2j+1} \int_{-\pi}^{\pi} \cdots \int d(\omega)_{2j+1}$$
$$\times K((\omega)_{2j+1})\left(1 + \cos \sum_{1}^{2j+1} \omega_k\right)\left[\cosh \sum_{1}^{2j+1} \gamma(\omega_k) - \cos \sum_{1}^{2j+1} \omega_k\right]^{-1} \tag{32}$$

Note that $\chi_{2j+1}(t) \geqslant 0$, so that any partial sum of (30) gives a lower bound to χ. In particular, from (22) with $n = 0$, it follows that

$$\chi_1(t) = F(\phi)^2 \int_{-\pi}^{\pi} d\omega/\sinh \gamma(\omega)\, [\cosh \gamma(\omega) - \cos \omega] \tag{33}$$

By making the change of variable $x = \omega/\gamma(0)$, (33) gives

$$\lim_{t\to 0} t^{7/4}\chi_1(t) = A \int_{-\infty}^{\infty} dx/(1 + x^2)^{1/2}(1 + 2x^2) \tag{34}$$

The same sort of reasoning implies that every term in (30) has the $t^{-7/4}$ divergence. Finally, it is clear that

$$\liminf_{t\to 0+} \log \chi(t, 0)/|\log t| \geqslant 7/4 \tag{35}$$

4. AN UPPER BOUND FOR $\chi(t, 0)$

By examining (22)–(26), we can rewrite (18) with $d\omega$ replaced by $d\omega/\sinh \gamma(\omega)$ and with $J((\omega)_{2n+1})$ replacing K, where

$$J((\omega)_{2n+1}) = |G^x((e^{i\omega})_{2n+1})|^2 \tag{36}$$

with

$$G^x((z)_{2n+1}) = \sum_{1}^{2n+1} (-1)^k G(\Delta_k(z)_{2n+1}) \tag{37}$$

In the above, G is given by a formula analogous to (26), but with contraction function

$$w_{-}(e^{i\omega_1}, e^{i\omega_2}) = [\sinh\gamma(\omega_1) - \sinh\gamma(\omega_2)]/\sin[(\omega_1 + \omega_2)/2] \tag{38}$$

By enumerating terms in the modified form of (18) and replacing each contraction by its supermum, one obtains

$$u_2(\mathbf{r}) \leqslant F(\phi)^2 \sum_0^{\infty} (\sup|w_{-}|)^{2j} \frac{[(2j+1)(2j-1)\cdots]^2}{(2j+1)!} I(r_1|\gamma(0))^{2j_1+1} \tag{39}$$

with

$$I(r|\gamma(0)) = (1/2\pi)\int_{-\pi}^{\pi} e^{-r\gamma(\omega)}\, d\omega/\sinh\gamma(\omega) \tag{40}$$

The combinatorial factor can be bounded above by $Aj^{1/2}$, where A is the Euler–Mascheroni constant. Thus (39) gives

$$u_2(\mathbf{r}) \leqslant F(\phi)^2 I(r_1|\gamma(0)) A \sum_0^{\infty} j^{1/2} a(t)^{2j} I(r_1|\gamma(0))^{2j} \tag{41}$$

which makes sense provided $a(t)I(r_1|\gamma(0)) < 1$, which is satisfied, for any $\gamma(0) > 0$, by $r_1 > s_0/\gamma(0)$, where s_0 is a constant. From (41) one obtains

$$\begin{aligned} B(s_0) &= \limsup_{t\to 0+} t^{7/4} \sum_{r_1 > s_0/\gamma(0)} r_1 u(r_1, 0) \\ &\leqslant B\int_{s_0}^{\infty} ds\, K_0{}^3(s)/[1 - K_0{}^2(s)]^2 \end{aligned} \tag{42}$$

where

$$K_0(s) = (1/2\pi)\int_{-\infty}^{\infty} e^{-s\cosh\theta}\, d\theta \tag{43}$$

is a Bessel function. For s_0 one may take any real number such that $K_0(s_0) < 1$.

The remaining problem is then to handle the rest of the sum in (6). Clearly

$$\limsup t^{7/4}\chi(t, 0) \leqslant A(s_0) + B(s_0) \tag{44}$$

where

$$A(s_0) = \limsup_{t\to 0+} t^{7/4} \sum_1^{s_0/\gamma(0)} r_1 u_2(r_1, 0) \tag{45}$$

Monotonicity in r_1 of $u(r_1, 0)$ gives

$$\begin{aligned} \sum_{r_1} r_1 u(r_1, 0) &\leqslant 2 \sum_{r_1\,\text{even}} (r_1 + 1) u(r_1, 0) \\ &\leqslant 2 \sum_{r_1\,\text{even}} (r_1 + 1) u(r_1/2, r_1/2) \end{aligned} \tag{46}$$

where the second inequality follows from the diagonal duplication argument of Messager and Miracle-Sole.[12] Now making the t dependence explicit, Griffiths' inequalities[16] give $u_2(\mathbf{r}|t) \leqslant u_2(\mathbf{r}|0)$ for $t \geqslant 0$. Onsager[13] has given $u(r, r|0)$ exactly. In fact,

$$u(r, r|0) = Cr^{-1/4}[1 + O(r^{-2})] \tag{47}$$

with $C = 0.645002448\ldots$. It follows that

$$C(s_0) \leqslant \int_0^{s_0} s^{3/4}\, ds \tag{48}$$

Using (42), (44), and (48), it follows from (6) that

$$\limsup_{t \to 0+} \log \chi(t, 0)/|\log t| \leqslant 7/4 \tag{49}$$

Thus the limit in (9) exists as $t \to 0+$, and moreover $\gamma = 7/4$. The same argument cannot be used for $t \leqslant 0$ because the Griffiths' inequality cannot be applied to $u_2(r)$ in a useful way.

Using Propositions 5.1, 5.2, and 5.3 of Ref. 1, we have the result

$$\chi(0, t) = |t|^{-7/4} \int_{s_0}^{\infty} ds\, sF_{\pm}(s) + \sum_{|\mathbf{r}| < s_0/\gamma(0)} u(\mathbf{r}) + C(t, s_0) \tag{50}$$

where $C(t, s_0)$ is the "correction to scaling" contribution to the fluctuation sum. Equation (50) is valid for any $s_0 > 0$. The sum in (50) is bounded above, using the previous Griffiths' inequality argument, by $|t|^{-7/4}$. Wu *et al.*[4] and, independently, Ryazanov[20] and Vaks *et al.*[21] have given an asymptotic estimate of $F_{\pm}(s)$ valid as $s \to 0$:

$$F_{\pm}(s) \sim F(0)/s^{1/4}$$

This ensures integrability of $sF_{\pm}(s)$ at $s = 0$.

5. THE CORRELATION LENGTH AND CRITICAL CORRELATION FUNCTION

Fisher[22] has defined a sequence of correlation lengths $\xi_\varphi(t, h)$ by

$$(\xi_\varphi(t, h))^{2\varphi} = \sum |\mathbf{r}|^{2\varphi} u_2(\mathbf{r}|t, h)/\chi(t, h) \tag{51}$$

Using the GHS technique of Ref. 5, it can be shown that $\xi_\varphi(t, h)$ is continuous in h at $h = 0$ for $t \neq 0$. Critical indices for the divergence when $h = 0$ of (51) as $t \to 0\pm$ are given by

$$\nu_\varphi = -\lim_{t \to 0+} \log \xi_\varphi(0, t)/\log|t| \tag{52}$$

with ν_φ' defined analogously when $t \to 0-$, assuming in both cases that the limit exists. Use of the upper and lower bounds given in Section 4 gives the result

$$\nu_{\varphi} = 1 \tag{53}$$

independent of φ. Fisher and Burford[14] have pointed out different definitions of the inverse correlation length and the difficulties which may arise in the naive interpretation of the inverse correlation length in terms of the mass gap, or spectral gap, of the transfer matrix. This was highlighted in the work of Johnson *et al.*[23] on the eight-vertex model. The "mass gap" definition is

$$\xi_{\infty}(t, h) = -\lim_{\mathbf{r}\to\infty} \log u_2(\mathbf{r}|t, h)/\log|\mathbf{r}| \tag{54}$$

Using the fact that the n-particle states make a nonnegative contribution to $u_2(\mathbf{r}|t, 0)$ in (18) for any $n \geqslant 1$, by considering $n = 1$ and the upper bound (28), it follows in a straightforward fashion that $\nu_{\infty} = 1$, once again by constructing upper and lower bounds in the definition (52).

Fisher has also defined the critical index η by

$$2 - \eta = \lim_{R\to\infty} \log \sum_{|\mathbf{r}|<R} u_2(\mathbf{r}|0, 0)/\log R \tag{55}$$

Once again we can use the duplication argument of Messager and Miracle-Sole[12] and Onsager's diagonal correlation function[13] to deduce that

$$2 - \eta \leqslant 7/4 \tag{56}$$

But, using the moment definition, (51), Fisher[22] has derived the exponent inequality for Ising ferromagnets

$$\gamma \leqslant (2 - \eta)\nu_1 \tag{57}$$

Consequently, introducing the rigorous results already obtained in this paper into (57) gives $2 - \eta \geqslant 7/4$ so that

$$\eta = 1/4 \tag{58}$$

Messager and Miracle-Sole[12] have also proved by duplication that since $u(r, r|0, 0) \sim A|r|^{-1/4}$, then the same form of asymptotic decay obtains in any direction.

ACKNOWLEDGMENTS

The author is most grateful to the Aspen Center for Physics, the IHES, and the Rockefeller University for providing hospitality while portions of this work were being carried out. He would like to acknowledge discussions with J. Groeneveld and B. Souillard and to thank A. Messager and S. Miracle-Sole for expounding their unpublished results on duplication.

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